Lecture Notes

Goal: Students understand how calculus (integration + differentiation) is used to model the exponential growth of populations.

Population Growth 6.5

How does a population grow?

exponentially, the more mommy + daddy rabbits you have the more baby rabbits that will be made

Biomathematicians do this all the time! Predator-Prey relationships, Climate Change, Endangered Species

Relative Growth

$$\frac{dP}{dt} = KP$$

The growth of population is proportional to current population.

K - "Growth Rate / Constant"

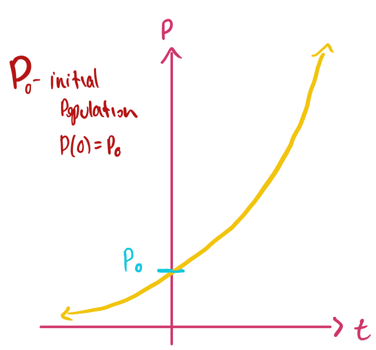
We need like terms on each

$$\int \frac{1}{P}\,dP = \int K\,dt$$

$$e^{(\ln|P|)} = e^{(Kt + C)}$$

$$P(t) = e^{Kt} \cdot e^{C}$$ Rename to P_0

$$\boxed{P(t) = P_0 e^{Kt}}$$



☆ This model assumes unlimited growth... what about Carrying Capacity???

Population

A long time ago in a galaxy far far away the total population was increasing at a rate of 1 person every 2hrs. At 5pm, the total population was 8,675,309

ⓐ What is the relative growth rate per day? Find K

$P(t) = P_0 e^{kt}$ We know $P_0 = P(0) = 8{,}675{,}309 \Rightarrow P(t) = 8{,}675{,}309\, e^{kt}$

We also know, after 2hrs (or $\frac{1}{12}$ days) $P(t) = P_0 + 1 = 8{,}675{,}310 \Rightarrow \underbrace{8{,}675{,}310}_{\text{add 1 person}} = 8{,}675{,}309\, e^{k \cdot \overbrace{\frac{1}{12}}^{\text{days it takes to add 1 person}}}$

Solve for $k \Rightarrow \ln\left(\frac{8{,}675{,}310}{8{,}675{,}309}\right) = \ln\left(e^{\frac{k}{12}}\right) \Rightarrow \boxed{\ln\left(\frac{8{,}675{,}310}{8{,}675{,}309}\right) \cdot 12 = k} = 1.3 \cdot 10^{-6}$

ⓑ What will the population be at 5pm tomorrow?

$P(1) = 8{,}675{,}309\, e^{1 \cdot \ln\left(\frac{8{,}675{,}310}{8{,}675{,}309}\right) \cdot 12} = 8{,}675{,}321$ people 12 more than originally

Disease Spread

We want to model the spread of a disease. The number of cases of the disease is reduced by 20% each year.

ⓐ If there are 10,000 today, how many years will it take to reduce the number to 1,000?

$8000 = 10000 e^{k} \Rightarrow \ln\left(\frac{8}{10}\right) = \underline{k = -0.22314} \Rightarrow P(t) = 10000 e^{-0.22314t}$

Find t $1000 = 10000 e^{-0.22314t} \Rightarrow \frac{\ln\left(\frac{1}{10}\right)}{-0.22314} = \boxed{t = 10.319 \text{ years}}$

ⓑ How long will it take to $\underbrace{\text{eradicate the disease}}_{P(t)=1}$?

$1 = 10000 e^{-0.22314t} \Rightarrow \boxed{t \approx 41 \text{ years}}$

Logistic Growth Model

aka "Exponential Growth with Carrying Capacity"

$$\boxed{\frac{dP}{dt} = kP\left(1-\frac{P}{M}\right) = kP - \frac{k}{M}P^2 = \frac{k}{M}P(M-P)}$$

where M - Carrying Capacity

$$\int \frac{1}{P(M-P)}\,dP = \int \frac{k}{M}\,dt$$ (Rewrite using partial fractions)

$$\int \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)dP = \frac{k}{M}t + C$$ (Rewriting)

$$\int \left(\frac{1}{P} + \frac{1}{M-P}\right)dP = kt + c$$ Integrate

$$\ln(P) - \ln(M-P) = kt + c$$ $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$

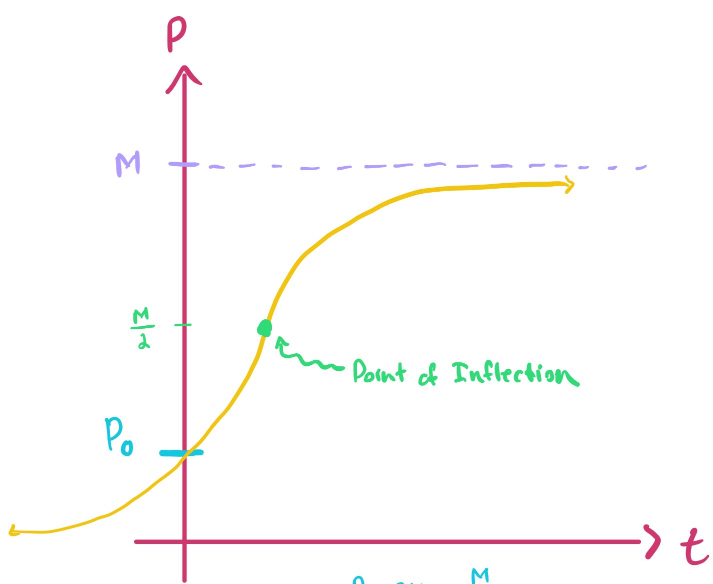
$$\ln\left(\frac{P}{M-P}\right) = kt + c$$ $\ln\left(\frac{a}{b}\right) = -\ln\left(\frac{b}{a}\right)$

$$-\ln\left(\frac{M-P}{P}\right) = kt + c$$ Exponentiate and distribute negative to other side

$$\frac{M-P}{P} = -e^{-kt-c}$$ Let $A = \pm e^{-c}$

$$\frac{M}{P} - 1 = Ae^{-kt}$$

☆ Population grows fastest where the slope is greatest: $\frac{M}{2}$ (The P.O.I.)



$P_0 = P(0) = \frac{M}{1+A}$

$$\boxed{P(t) = \frac{M}{1+Ae^{-kt}}}$$ "Logistic Regression Equation"

ex] let $k = 0.05$, $M = 200$, $P(0) = 10$

ⓐ logistic Growth equation

$$P(t) = \frac{200}{1 + Ae^{-0.05t}}$$

$$10 = \frac{200}{1 + A} \Rightarrow 10 + 10A = 200$$

$$A = 19$$

Thus, $$\boxed{P(t) = \frac{200}{1 + 19e^{-0.05t}}}$$

Gorilla Population

A wildlife preserve can support no more than 250 gorillas. 28 gorillas were in the preserve in 1970. Rate of population growth is

$$\frac{dp}{dt} = 0.0004P(250 - P)$$ t measured in years

ⓐ Find formula for gorilla population in terms of t

$$\frac{dp}{dt} = 0.0004P(250 - P) \Rightarrow \int \frac{1}{P(250 - P)}\, dp = \int 0.0004\, dt$$

$\frac{A}{P} + \frac{B}{250 - P} = 1$

$A(250 - P) + BP = 1$

$A = B = \frac{1}{250}$

By Partial fractions

$$\int \frac{1}{250}\left[\frac{1}{P} + \frac{1}{250 - P}\right] dp = 0.0004t + C$$

$$\ln(P) - \ln(250 - P) = 0.1t + C$$

$$\ln\left(\frac{P}{250 - P}\right) = 0.1t + C \Rightarrow e^{\ln\left(\frac{250 - P}{P}\right)} = e^{(-0.1t - C)}$$

$$\Rightarrow \frac{250}{P} - 1 = Ae^{-0.1t}$$

$$\Rightarrow P(t) = \frac{250}{1 + Ae^{-0.1t}}$$

Find A

$P(0) = 28$ ✓

$$28 = \frac{250}{1 + A} \Rightarrow A = \frac{222}{28} = 7.928$$

$$\boxed{P(t) = \frac{250}{1 + 7.928e^{-0.1t}}}$$

★ **Alternative Route** : We know: $\frac{dp}{dt} = \frac{k}{M} P(M-P)$

Thus, $M = 250$

and $\frac{k}{250} = 0.0004 \Rightarrow k = 0.1$

$P(t) = \frac{250}{1 + Ae^{-0.1t}}$ $P(0) = 28$

$28 = \frac{250}{1 + A} \Rightarrow A = 7.928 \Rightarrow \boxed{P(t) = \frac{250}{1 + 7.928e^{-0.1t}}}$

(b) How long will it take to reach carrying capacity?

$249.5 = \frac{250}{1 + 7.928e^{-0.1t}}$

Solve for t: $249.5 + 249.5 \cdot 7.928e^{-0.1t} = 250$

$e^{-0.1t} = \frac{0.5}{249.5} \cdot \frac{1}{7.928}$

$-0.1t = \ln(0.0002527)$

$t = -10 \cdot \ln(0.0002527)$

$\boxed{t \approx 83 \text{ years}}$

Practice Problems

Bear Ex | Given: $M=100$, $P_0=10$, $k=0.10$. Find $P(t)=?$

$$\int \frac{dP}{dt} = 0.10 \cdot P\left(1-\frac{P}{100}\right) = \int \frac{1}{1000} \cdot P(100-P)$$

$$\int \frac{1}{P(100-P)}\,dP = \int \frac{1}{1000}\,dt$$

$$\frac{A}{P} + \frac{B}{100-P} = A(100-P) + B(P) = 1$$

$$B = A = \frac{1}{100}$$

$$\frac{1}{100}\left[\int \frac{1}{P} + \int \frac{1}{100-P}\right] = \frac{1}{1000}\int dt$$

$$\ln|P| + \ln|100-P| = \frac{t}{10} + C$$

$$\ln\left|\frac{P}{100-P}\right| = \frac{t}{10} + C \Rightarrow \ln\left|\frac{100-P}{P}\right| = -\frac{t}{10} + C$$

* Distribute Negative to $\frac{t}{10}$

$$\Rightarrow \frac{100-P}{P} = Ae^{-t/10}$$

$$\Rightarrow \frac{100}{P} - 1 = Ae^{-t/10} \Rightarrow \frac{100}{P} = 1 + Ae^{-t/10}$$

$$\boxed{P(t) = \frac{100}{1+Ae^{-t/10}}}$$

We Know $P(0)=10$

$$10 = \frac{100}{1+A} \Rightarrow 10(1+A) = 100$$

$$1+A = 10$$

$$A = 9$$

$$\Rightarrow \boxed{P(t) = \frac{100}{1+9e^{-\frac{t}{10}}}}$$

When will $P=50$?

$$50 = \frac{100}{1+9e^{-\frac{t}{10}}} \Rightarrow 50\left(1+9e^{-\frac{t}{10}}\right) = 100$$

$$e^{-\frac{t}{10}} = \frac{1}{9} \Rightarrow -\frac{t}{10} = \ln\left|\frac{1}{9}\right|$$

$$\Rightarrow t = -10 \cdot \ln\left|\frac{1}{9}\right| \approx \boxed{22 \text{ years}}$$

Bacteria If there are 200 bacteria after 2hrs and 800 bacteria after 5hrs.

ⓐ How many bacteria were present initially? Find P_0

Given: $\begin{cases} 200 = P_0 e^{2k} \longrightarrow P_0 = \dfrac{200}{e^{2k}} \\ 800 = P_0 e^{5k} \end{cases}$

plug in

$\Rightarrow 800 = \dfrac{200e^{5k}}{e^{2k}} \Rightarrow 4 = e^{3k} \Rightarrow \underbrace{k = \dfrac{\ln(4)}{3}}_{\text{Plug back into}}$

$$\Rightarrow P_0 = \frac{200}{e^{2k}} = \frac{200}{e^{2\cdot\frac{\ln(4)}{3}}} = \boxed{79.37 \text{ bacteria}}$$

ⓑ After how many hours will the bacteria be 50,000?

$P(t) = 79.37e^{\frac{\ln(4)}{3}\cdot t}$ $\quad 50{,}000 = 79.37e^{\frac{\ln(4)}{3}\cdot t}$ Solve for t

$$t = \ln\left(\frac{50{,}000}{79.37}\right)\cdot\frac{3}{\ln(4)} = \boxed{13.9 \text{ years}}$$

Name Key Date Period

7.4 **Independent Practice—Logistic Growth**
Show all work. No calculator unless stated.

Multiple Choice

1. The spread of a disease through a community can be modeled with the logistic equation $y = \frac{600}{1+59e^{-0.1t}}$, where y is the number of people infected after t days. How many people are infected when the disease is spreading the fastest?

   (A) 10 (B) 59 (C) 60 (D) 300 (E) 600

$600 \to L$

$\frac{600}{2}$

2. The spread of a disease through a community can be modeled with the logistic equation $y = \frac{0.9}{1+45e^{-0.15t}}$, where y is the proportion of people infected after t days. According to the model, what percentage of people in the community will not become infected?

   (A) 2% (B) 10% (C) 15% (D) 45% (E) 90%

Carrying capacity → max 0.9 → 90% infected

∴ 10% will not be infected

3. $\displaystyle\int_2^3 \frac{3}{(x-1)(x+2)}\,dx = \int_2^3 \left(\frac{1}{x-1} + \frac{-1}{x+2}\right)dx$

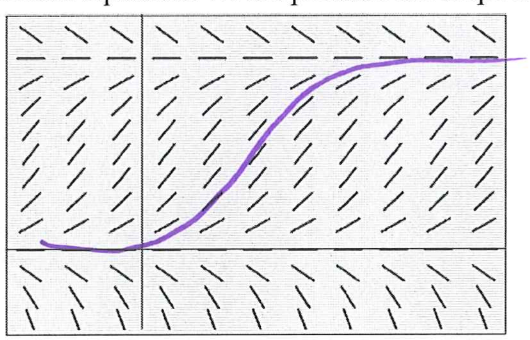
(A) $-\dfrac{33}{20}$ (B) $-\dfrac{9}{20}$ (C) $\ln\left(\dfrac{5}{2}\right)$ (D) $\ln\left(\dfrac{8}{5}\right)$ (E) $\ln\left(\dfrac{2}{5}\right)$

$= \ln\left|\dfrac{x-1}{x+2}\right| \Big]_2^3$

$= \ln\left|\dfrac{2}{5}\right| - \ln\left|\dfrac{1}{4}\right|$

$= \ln\left(\dfrac{2}{5}\cdot\dfrac{4}{1}\right) = \dfrac{8}{5}$

4. Which of the following differential equations would produce the slope field shown below?



[–3, 8] by [–50, 150]

logistic shape $\Rightarrow \dfrac{dy}{dx} = ky(L-y)$

(A) $\dfrac{dy}{dx} = 0.01x(120-x)$ (B) $\dfrac{dy}{dx} = 0.01y(120-y)$ (C) $\dfrac{dy}{dx} = 0.01y(100-x)$

(D) $\dfrac{dy}{dx} = \dfrac{120}{1+60e^{-1.2x}}$ (E) $\dfrac{dy}{dx} = \dfrac{120}{1+60e^{-1.2y}}$

$y =$
not dy/dx

5. The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, where the initial population is $P(0) = 3000$ and t is the time in years. What is $\lim_{t \to \infty} P(t)$?

(A) 2500   (B) 3000   (C) 4200   (D) 5000   (E) 10,000

$$\frac{dP}{dt} = \frac{1}{5000}P(\underbrace{10{,}000}_{L} - P)$$

6. Suppose a population of wolves grows according to the logistic differential equation $\frac{dP}{dt} = 3P - 0.01P^2$, where P is the number of wolves at time t, in years. Which of the following statements are true?

I. $\lim_{t \to \infty} P(t) = 300$ ✓ (carrying capacity)

II. The growth rate of the wolf population is greatest when $P = 150$. ✓ $\frac{300}{2}$

III. If $P > 300$, the population of wolves is increasing. ✗ False, dec towards 300

(A) I only   (B) II only   (C) I and II only   (D) II and III only   (E) I, II, and III

$$\frac{dP}{dt} = 0.01P(\underbrace{300}_{L} - P)$$

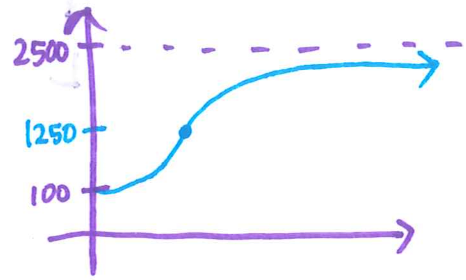
Short Answer/Free Response

Work the following on notebook paper.

7. Suppose the population of bears in a national park grows according to the logistic differential equation $\frac{dP}{dt} = 5P - 0.002P^2$, where P is the number of bears at time t in years.

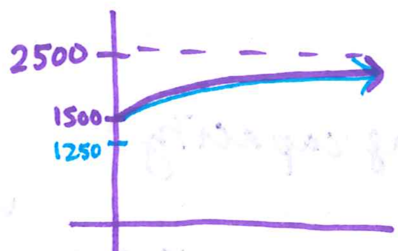
   (a) If $P(0) = 100$, then $\lim_{t\to\infty} P(t) =$ 2500. Sketch the graph of $P(t)$. For what values of P is the graph of P increasing? decreasing? Justify your answer.

   $\frac{dP}{dt} = 0.002P(\underset{\uparrow \atop L}{2500} - P)$

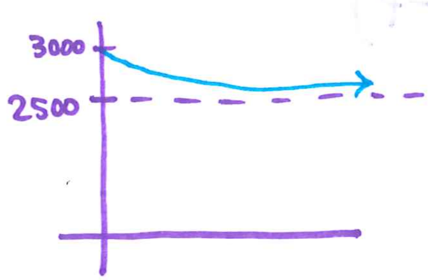
   f(t) is increasing for ~~(100, 2500)~~ $100 < P < 2500$

   (b) If $P(0) = 1500$, $\lim_{t\to\infty} P(t) =$ 2500. Sketch the graph of $P(t)$. For what values of P is the graph of P increasing? decreasing? Justify your answer.

   f(t) is increasing for $1500 < P < 2500$

   (c) If $P(0) = 3000$, $\lim_{t\to\infty} P(t) =$ 2500. Sketch the graph of $P(t)$. For what values of P is the graph of P increasing? decreasing? Justify your answer.

   f(t) is decreasing for $2500 < P < 3000$

   (d) How many bears are in the park when the population of bears is growing the fastest? Justify your answer.

   $P(t) = \frac{2500}{2} = 1250$ bears, the location of the inflection point

8. (Calculator Permitted) A population of animals is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = 0.01P(100 - P)$, where t is measured in years.

k L

(a) If $P(0) = 20$, solve for P as a function of t.

$$y = \frac{100}{1 + Ce^{-100 \cdot .001 \cdot t}}$$

$$y = \frac{100}{1 + Ce^{-t}}$$

$$\boxed{P(t) = \frac{100}{1 + 4e^{-t}}}$$

$$20 = \frac{100}{1 + Ce^{0}}$$

$$1 + c = \frac{100}{20}$$

$$1 + c = 5 \rightarrow c = 4$$

(b) Use your answer to (a) to find P when $t = 3$ years. Give exact and 3-decimal approximation.

$$P(3) = \frac{100}{1 + 4e^{-3}} = \boxed{83.393 \text{ animals}}$$

(c) Use your answer to (a) to find t when $P = 80$ animals. Give exact and 3-decimal approximation.

$$P(t) = 80$$

$$80 = \frac{100}{1 + 4e^{-t}}$$

$$1 + 4e^{-t} = \frac{100}{8} = \frac{5}{4}$$

$$\boxed{t = \ln 16 \text{ years}}$$

$$= 2.772 \text{ years}$$

(calculator)

$$4e^{-t} = \frac{1}{4}$$

$$e^{-t} = \frac{1}{16}$$

$$-t = \ln(1/16)$$

(no calc)

$$t = -\ln(1/16) = \boxed{\ln 16 \text{ years}}$$

9. (Calculator Permitted) The rate at which a rumor spreads through a high school of 2000 students can be modeled by the differential equation $\frac{dP}{dt} = 0.003P(2000 - P)$, where P is the number of students who have heard the rumor t hours after 9AM.

(a) How many students have heard the rumor when it is spreading the fastest?

$\frac{2000}{2} =$ 1000 students

(b) If $P(0) = 5$, solve for P as a function of t.

$P(t) = \frac{2000}{1+Ce^{-6t}}$

$5 = \frac{2000}{1+Ce^{0}}$

$1+C = \frac{2000}{5} = 400 \rightarrow C = 399$

$P(t) = \frac{2000}{1+399e^{-6t}}$

(c) Use your answer to (b) to determine how many hours have passed when the rumor is spreading the fastest. Give exact and 3-decimal approximation.

$P(t) = 1000$

$t = 0.998$ hrs (calculator)

$1000 = \frac{2000}{1+399e^{-6t}}$

$1+399e^{-6t} = 2$

$399e^{-6t} = 1$

$e^{-6t} = \frac{1}{399}$

$-6t = \ln\left(\frac{1}{399}\right)$

$t = \frac{1}{6}\ln(399)$ (no calc)

(d) Use your answer to (b) to determine the number of people who have heard the rumor after two hours. Give exact and 3-decimal approximation.

$P(2) = \frac{2000}{1+399e^{-12}} = 1995.108$ students

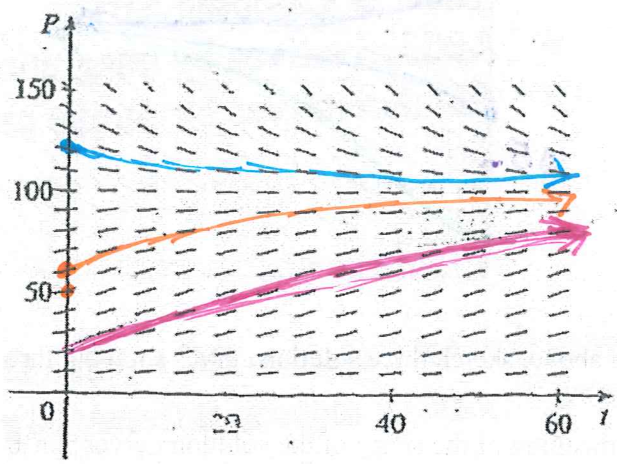
10. Suppose that a population develops according to the logistic equation $\frac{dP}{dt} = 0.05P - 0.0005P^2$ where t is measured in weeks.

(a) What is the carrying capacity/limit to growth?

100

$0.0005P(100-P)$

(b) A slope field for this equation is shown below.



I. Where are the slopes close to zero?

Near $P=0$ and $P=100$

II. Where are they largest?

$P=50$

III. Which solutions are increasing?

$0 < P < 100$

IV. Which solutions are decreasing?

$100 < P < 150$

(c) Use the slope field to sketch solutions for initial populations of 20, 60, and 120.

I. What do these solutions have in common?

$\lim_{t \to \infty} P(t) = 100$

II. How do they differ?

growth rates; initial points; only first has inf pt'

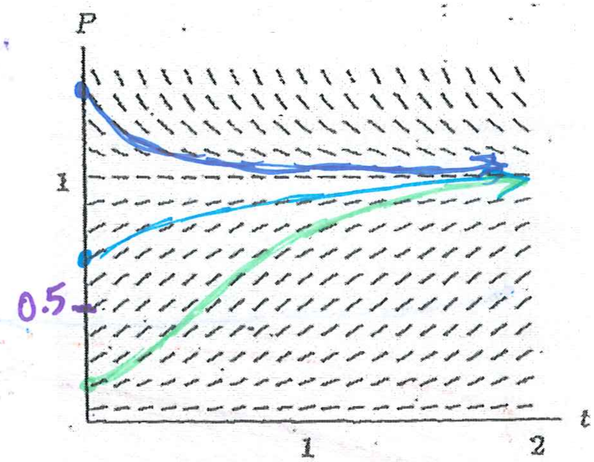
III. Which solutions have inflection points?

Only $P(0)=20$

IV. At what population level do these inflection points occur?

At $P=50$

11. The slope field show below gives general solutions for the differential equation given by $\frac{dP}{dt} = 3P - 3P^2$.



(a) On the graph above, sketch three solution curves showing three different types of behavior for the population P.

(b) Describe the meaning of the shape of the solution curves for the population.

I. Where is P increasing?

from P=0 to P=1

II. Where is P decreasing?

above P=1

III. What happens in the long run (for large values of t)?

they approach P=1

IV. Are there any inflection points? If so, where?

There is an inf pt when P=0.5

V. What do the inflection points mean for the population?

The population is growing fastest at this point

Question 5

A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

(a) If $P(0) = 3$, what is $\lim_{t\to\infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t\to\infty} P(t)$?

(b) If $P(0) = 3$, for what value of P is the population growing the fastest?

(c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function Y found in part (c), what is $\lim_{t\to\infty} Y(t)$?

(a) For this logistic differential equation, the carrying capacity is 12. If $P(0) = 3$, $\lim_{t\to\infty} P(t) = 12$. If $P(0) = 20$, $\lim_{t\to\infty} P(t) = 12$.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{answer} \end{cases}$
(b) The population is growing the fastest when P is half the carrying capacity. Therefore, P is growing the fastest when $P = 6$.	1 : answer
(c) $\frac{1}{Y}dY = \frac{1}{5}\left(1 - \frac{t}{12}\right)dt = \left(\frac{1}{5} - \frac{t}{60}\right)dt$ $\ln\lvert Y\rvert = \frac{t}{5} - \frac{t^2}{120} + C$ $Y(t) = Ke^{\frac{t}{5} - \frac{t^2}{120}}$ $K = 3$ $Y(t) = 3e^{\frac{t}{5} - \frac{t^2}{120}}$	$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } Y \\ \quad 0/1 \text{ if } Y \text{ is not exponential} \end{cases}$ Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables
(d) $\lim_{t\to\infty} Y(t) = 0$	1 : answer $0/1$ if Y is not exponential

Copyright © 2004 by College Entrance Examination Board. All rights reserved.
Visit apcentral.collegeboard.com (for AP professionals) and www.collegeboard.com/apstudents (for AP students and parents).